Equilibrium analysis of game on heterogeneous networks with coupled activities

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Abstract—We study a game where agents interacting over a network engage in two coupled activities and have to strategically decide their production for each of these activities. Agent interactions involve local and global network effects, as well as a coupling between activities. We consider the general case where the network effects are heterogeneous across activities, i.e., the underlying graph structures of the two activities differ and/or the parameters of the network effects are not equal. In particular, we apply this game in the context of palm oil tree cultivation and timber harvesting, where network structures are defined by spatial boundaries of concessions. We first derive a sufficient condition for the existence and uniqueness of a Nash equilibrium. This condition can be derived using the potential game property of our game or by employing variational inequality framework. We show that the equilibrium can be expressed as a linear combination of two Bonacich centrality vectors.

I. INTRODUCTION

We study interactions between economic agents who are simultaneously engaged in the production of multiple goods and compete in a market to sell these goods. In many situations, such trade relationships are described by a network structure that captures how the aggregate production of each good is influenced by the manner in which each agent is connected with other agents. When such network connections are heterogeneous across goods, their impact on the agents’ utility need to be modeled separately. Often, the production levels of goods (henceforth, referred as activities) are coupled because of the underlying complementarity or substitutability effects.

For example, palm oil tree cultivation and timber harvesting from forest concessions in the tropical regions of Southeast Asia are inherently coupled activities [1, 2, 3]. Here, the incentives of individual agents (palm oil and logging companies) are not only shaped by the spatial distribution (i.e., network structure) of timber and logging concessions, but also depend on how these activities are coupled. One can argue that in this example the coupling depends on the extent to which the activities can be carried in a synergistic manner (e.g., by using similar means of production and transport of harvested goods [4]) or compete with each other in terms of resources (e.g., water, sunlight, and soil nutrients and/or labor and capital [5, 6, 7]). The competition for resources and economic outlet arises also at the global level between agents, giving rise to a global network effect besides a local network effect.

In this paper, we study a network game in which the activities (i.e., production decisions) of each agent is influenced by her interactions with other agents in the network, as well as the coupling between these activities. Importantly, the network interactions corresponding to each activity can be heterogeneous and coupling vary across agents. In this sense, our game-theoretic approach extends the well-known network games with single activity [8, 9, 10] and multiple activities [11, 12].

In [8, 9, 10, 13], the agent utility is a linear-quadratic function given by \( u(y_i, y_{-i}, G) = py_i - \frac{1}{2}cy_i^2 - \mu \sum_{j=1}^n y_j + \delta \sum_{j=1}^n G_{ij}y_iy_j \), where \( G \) is the adjacency matrix of the graph \( G(\mathcal{N}, \mathcal{E}) \) underlying the network structure, where the set of nodes of the graph \( \mathcal{N} \), with \( |\mathcal{N}| = n \), models agents and the set of edges \( \mathcal{E} \) represent their interactions. For any node \( i \), the neighborhood of \( i \) is the set of nodes \( j \) connected to \( i \) by an edge, i.e., \( G_{ij} = 1 \). Furthermore, \( y_i \) is the production of agent \( i \), \( y_{-i} \) is the production of all agents except \( i \), \( p \) is the price of the commodity, \( c \) is the cost of production, \( \mu \) is the parameter quantifying the global network effect due to the competition of agents in a market to sell their productions, \( \delta \) is the parameter quantifying the local network effect that arises from the interaction of agents with their neighbors. While [8] considered this model in the context of criminal network, [9, 10, 13] applied it to education, R&D and financial risk. Our focus is on production networks and our context pertains to harvesting and trade of coupled forest products (e.g., timber and palm oil) where network effects arise from spatial connections between...
forest regions and concessions, and coupling arises from other aspects such as availability of natural resources, production technology and transportation routes [4, 5].

Previous works on network games with single activity [8, 10, 13] have shown that provided the local network effect is small enough compared to the parameter for own concavity $c$, namely $\delta \lambda_{\text{max}}(G) < c$, where $\lambda_{\text{max}}$ is the largest eigenvalue of $G$, then a unique Nash equilibrium exists and the production of individual agents at the equilibrium can be expressed in terms of the vector of Bonacich centralities, a canonical centrality measure on network.

**Definition 1.** For a graph with adjacency matrix $G$ and scalar $\delta > 0$, let $M(G, \delta) = (I - \delta G)^{-1} = \sum_{k=0}^{\infty} \delta^k G^k$. Given a vector of weights $w \in \mathbb{R}^n_+$, the vector of weighted Bonacich centralities for the network $G$ is defined as

$$B_w(G, \delta) = M(G, \delta)w = (I - \delta G)^{-1}w = \sum_{k=0}^{\infty} \delta^k G^k w.$$ 

When $w = 1$, we simply denote $B_1 = B$. Then the unweighted Bonacich centrality of node $i$ is given by $B_i(G, \delta) = \sum_{j=1}^{n} M_{ij}(G, \delta)$, where $M_{ij}$ is the $(i,j)$-th entry of matrix $M$. As $G^k$ counts the number of paths of length $k$ from node $i$ to node $j$, the Bonacich centrality of node $i$ counts the total number of walks that start at node $i$ in the graph with adjacency matrix $G$. Each walk is exponentially discounted by $\delta$, i.e., longer walks have a lower weight in the centrality measure than shorter walks. In the context of trade, this discounting captures reduced impact of nodes (agents) that are connected via longer trading routes - with distances counted as number of hops.

We can express the (unweighted) Bonacich centrality as follows:

$$B_i(G, \delta) = M_{ii}(G, \delta) + \sum_{i \neq j} M_{ij}(G, \delta),$$

where $M_{ii}(G, \delta)$ counts the number of self-loops from $i$ to itself and $\sum_{i \neq j} M_{ij}(G, \delta)$ counts the number of outer walks from $i$ to any other player $j \neq i$. By definition $M_{ii}(G, \delta) \geq 1$, hence $B_i(G, \delta) \geq 1$, with equality when $\delta = 1$.

A particularly relevant work to ours is [11] in which the authors considered a game where agents engage in two coupled activities with homogeneous network effects (i.e., when both activities are subject to identical network effects). In this game, agent’s utility is given by $u(y_i^A, y_i^B, p^A, p^B) = p_i^A y_i^A - \frac{1}{2} c(y_i^A)^2 + \delta \sum_{j=1}^{n} G_{ij} y_j^A y_i^A + p_i^B y_i^B - \frac{1}{2} c(y_i^B)^2 + \delta \sum_{j=1}^{n} G_{ij} y_j^B y_i^B + \beta y_i^A y_i^B$, where $y_i^A$ and $y_i^B$ are the production of agent $i$ in activity $A$ and $B$ respectively, $\delta^A$ and $\delta^B$ are the local network effect for each activity, and $\beta$ is the parameter for the coupling effect, measuring the interdependence between activities.

In the setting of [11], the networks for activities $A$ and $B$ are assumed to be the same, i.e., the adjacency matrix $G$ encodes for the structure of the underlying graph. Furthermore, the local network effect encoded by the parameter $\delta$ is also identical for the two activities. The authors show that in equilibrium, production of each activity can be described by the sum of two terms, where each term is a weighted Bonacich centrality associated with the adjacency matrix $G$. In each of these terms, the discount factor $\delta$ is determined by the level of coupling $\beta$ between activities, and weights depend on the price vectors $p^A = [p_1^A, \ldots, p_n^A]^T$ and $p^B = [p_1^B, \ldots, p_n^B]^T$. Importantly, when the two activities are coupled, the condition for existence and uniqueness of an equilibrium is tighter, namely $\delta \lambda_{\text{max}}(G) < c - |\beta|$. It is easy to see that for the uncoupled case ($\beta = 0$), the equilibrium level of each activity is given by a Bonacich centrality associated with the adjacency matrix $G$, with discount factor $\delta$, and weight vectors $p^A$ and $p^B$ for activity $A$ and $B$ respectively.

In this paper, we extend the work of [11] in two directions: firstly, we consider heterogeneous network effects wherein an agent’s interaction with other agents for each activity is described by a different network structure, and/or the local network effect $\delta$ is different across activities. This allows us to capture more realistic situations such as that of harvesting from timber concessions and palm oil plantations, where the spatial configurations of forest concessions and plantations are described by different network structures and the production of the two activities are coupled. In particular, we consider agent-specific coupling parameter $\beta_i$, a positive parameter $\beta_i$ expresses the complementarity of activities $A$ and $B$, for instance because of common technologies, shared transportation and/or supply chains. By contrast, a negative parameter $\beta_i$ means that activities $A$ and $B$ are substitutes, for instance because of resources competition (groundwater, land, nutrients, sunlight).

Secondly, we also consider a global network effect for each activity (via fully connected network) to model the competition that agents face in selling the produced goods in a market (for example, global market of timber and palm oil products).

Our technical contributions are as follows: We show that the condition for the existence and uniqueness of Nash equilibrium can be derived by analyzing the potential game formulation, or by leveraging the results on variational inequality for equilibrium. In Section III we derive this condition for our network game with coupled activities (Theorem 1) by leveraging a preliminary result (Lemma 1 in Section II). Furthermore, we show that the

1 The coupling effect can modulate the local network effects since a large parameter $\beta_i$ in absolute value will affect both the parameters $\delta^A$ and $\delta^B$, and the structures of the networks.
equilibrium can be expressed as a linear combination of two Bonacich centralities for our general network game with heterogeneous local network effects and presence of global competition among agents (Theorem 2). To further interpret our results, we conduct numerical results with heterogeneous local network effects and presence of two Bonacich centralities for our general network game. The equilibrium can be expressed as a linear combination of agents’ actions.

For all players except $i$, the action profile $y$ is the set of actions, $Y_i$ is the set of available actions for agent $i$, and $u_i$ is the agent individual utility function. We define by $Y = Y_1 \times \cdots \times Y_n$ the set of all action profiles. Each agent’s action is multi-dimensional and denoted by $y_i \in \mathbb{R}^N$, where $N$ is the number of activities. For simplicity, we will limit our attention to $N = 2$. We also define $y_{-i}$ the action profiles for all players except $i$, and $y = (y_i, y_{-i})$ the action profiles for all players. The objective of each agent is to maximize her utility. An action profile $y \in \mathbb{R}^{nN}$ is called a Nash equilibrium (NE) if no agent has an incentive to unilaterally change her strategy.

Definition 2. A pure strategy Nash equilibrium is a profile of actions $y \in Y = Y_1 \times \cdots \times Y_n$ such that for all $i \in N$:

$$u_i(y_i, y_{-i}) \geq u_i(\tilde{y}_i, y_{-i}); \forall \tilde{y}_i \in Y_i.$$ 

It turns out that our network game $\Gamma$ is a potential game (See Section III).

Definition 3. A game $\Gamma$ is a potential game if there exists a function $\Phi : Y \to \mathbb{R}$ such that $\forall i \in n, \forall y_{-i} \in Y_{-i}, \forall y_i, \tilde{y}_i \in Y_i$, we have

$$\Phi(y_i, y_{-i}) - \Phi(\tilde{y}_i, y_{-i}) = u_i(y_i, y_{-i}) - u_i(\tilde{y}_i, y_{-i}).$$

The function $\Phi$ is called the potential function of the game $\Gamma$.

By [9] and [14], a profile of action $y$ is a Nash equilibrium of $\Gamma$ if and only if $y$ satisfies the Kuhn-Tucker conditions of the problem

$$\max_{y \in Y} \Phi(y, y_{-i}) .$$

Each agent chooses its production as if she wanted to maximize the potential function, given other agents’ production. This maximization problem has a unique solution when there is only one solution to its first-order conditions. For $y_i \geq 0, \forall i$, a sufficient condition for a unique solution is for the potential function $\Phi(y_i, y_{-i})$ to be strictly concave, i.e., the negative of the Hessian matrix of $\Phi(y_i, y_{-i})$ is positive definite, $-H > 0$.

As an alternative to the potential function approach, one can use the variational inequality framework [12, 15, 16] to study existence and uniqueness of network games.

Assumption 4. For all $i \in \{1, \cdots, n\}$, set $Y_i$ is non-empty, closed and convex, and the utility function $u_i(y_i, y_{-i})$ is continuously differentiable and convex in $y_i$ and for all $y_j \in Y_j$, $\forall j \neq i$ in the neighborhood of $i$. Furthermore, the utility function is twice differentiable in $[y_i, z_i]$, and $\nabla_{y_i} u_i(y_i, y_{-i})$ is Lipschitz in $[y_i, z_i]$, where $z_i = \left[ \sum_{j=1}^{n} G_{ij} y_j^N \right]_{\chi=1} \in \mathbb{R}^N$.

Definition 5. A vector $y \in \mathbb{R}^{nN}$ solves the variational inequality VI$(Y, V)$ with set $Y = Y_1 \times \cdots \times Y_n$ and operator $V : Y \to \mathbb{R}^{nN}$ if and only if

$$V(\hat{y})^T (y - \hat{y}) \geq 0, \forall y \in Y. \quad (1)$$

Under Assumption 4 [15, Proposition 1.4.2] show that $y$ is a Nash equilibrium for game $\Gamma$ if and only if it solves the variational inequality VI$(Y, V)$. Furthermore, in [15, Theorem 2.3.3], the authors show that the variational inequality VI$(Y, V)$ (1), where $V$ is continuous and $Y$ is nonempty, closed and convex, admits a unique solution if $V$ is strongly monotone.

Definition 6. An operator $V$ is strongly monotone if there exists $\alpha > 0, \alpha \in \mathbb{R}$, such that

$$\left( V(y) - V(\hat{y}) \right)^T (y - \hat{y}) \geq \alpha \|y - \hat{y}\|_2^2,$$

for all $\hat{y}, y \in Y$.

It follows that for $y$ to be a Nash equilibrium of game $\Gamma$, it is necessary and sufficient to prove that it solves the variational inequality VI$(Y, V)$; moreover a sufficient condition to prove the uniqueness of equilibrium is to show that the operator $V$ is strongly monotone.

If $V$ is an affine map, strong monotonicity is equivalent to strict monotonicity, i.e., $\left( V(y) - V(\hat{y}) \right)^T (y - \hat{y}) \geq 0, \forall \hat{y}, y \in Y$. A sufficient condition for $V$ to be strictly monotone is to show that its Jacobian $\nabla_{y} V(y)$ is positive definite [15, Proposition 2.3.2].

In fact, [14, Lemma 4.4] show that a game is a potential game with potential function $\Phi$ if and only if $\nabla_{y} \Phi(y) = [\nabla_{y} u_i(y_i, y_{-i})^T]_{i=1}^n$. We define $V(y) := -\nabla_{y} \Phi(y) = [-\nabla_{y} u_i(y_i, y_{-i})^T]_{i=1}^n$. Then, the first-order optimality conditions of the optimization problem $\max_{y \in Y} \Phi(y)$ can be expressed as $\nabla_{y} \Phi(y)(y - \hat{y}) \geq 0$, which coincide with the variational inequality (1) $V(y)^T (y - \hat{y}) \geq 0, \forall y \in Y$. Consequently, we can establish the equivalence between strict concavity of $\Phi$ and strong monotonicity of $V(y) := -\nabla_{y} \Phi(y)$ as condition for uniqueness of Nash equilibrium.

Lemma 1. A game $\Gamma$ with utility function $u_i(y_i, y_{-i}), i \in \{1, \cdots n\}$ is a potential game with potential function $\Phi(y)$ if and only if
\( \nabla_y \Phi(y) = -V(y) = [\nabla_y u_i(y, y)^\top]_{i=1}^n \). Then, there exists a unique Nash equilibrium if \( \Phi \) is strictly concave (equivalently if \( V \) is strongly monotone).

**Proof.** From [9], strict concavity of \( \Phi \) guarantees that the Nash equilibrium is unique. By definition, \( \Phi \) is strictly concave if and only if \( (\nabla_y \Phi(y) - \nabla_y \Phi(\bar{y}))^\top (\bar{y} - y) > 0, \forall \bar{y}, y \in \mathbb{Y} \); that is, there exists \( \alpha \in \mathbb{R}, \alpha > 0 \), such that \( (\nabla_y \Phi(y) - \nabla_y \Phi(\bar{y}))^\top (\bar{y} - y) \geq \alpha \| \bar{y} - y \|^2 \). Equivalently, the Nash equilibrium of \( \Gamma \) is unique if and only if \( -\nabla_y \Phi(y) \) is strongly monotone. The strict concavity of \( \Phi \) is equivalent to the strong monotonicity of \( V \).

**III. Existence, Uniqueness and Equilibrium Characterization**

In this section, we specify our network game \( \Gamma \) and derive a condition for the existence and uniqueness of NE, which we then characterize.

**A. Model**

In our model, \( n \) agents interact over a network. Each agent’s decision is her levels of production of two coupled activities denoted by \( A \) and \( B \). Let \( G^A \) and \( G^B \) denote the adjacency matrices for the network influencing activities \( A \) and \( B \), respectively. Each agent corresponds to a single node in the graph \( G^A(N, E^A) \) and the graph \( G^B(N, E^B) \), where \( N \) is the set of nodes and \( E^A \) (resp. \( N^B, E^B \)) is the set of edges in the network \( A \) (resp. \( B \)). For any node \( i \), the neighborhood of \( i \) in the network \( A \) (resp. \( B \)) is the set of nodes \( j \) connected to \( i \) by an edge, i.e., \( G^A_{ij} = 1 \) (resp. \( G^B_{ij} = 1 \)). Each agent chooses a level of production for activities \( A \) and \( B \), denoted by \( y^A_i \) and \( y^B_i \) respectively, when \( y^A_i \) and \( y^B_i \) are non-negative. Each agent’s action is thus two-dimensional. Let us denote \( y_i = [y_i^A, y_i^B]^\top \), \( y^A = [y_1^A, \ldots, y_n^A]^\top \), \( y^B = [y_1^B, \ldots, y_n^B]^\top \), and \( y = [y^A, y^B]^\top \). We also define \( y_{-i} = [y_1^A, \ldots, y_{i-1}^A, y_{i+1}^A, \ldots, y_n^A, y_1^B, \ldots, y_{i-1}^B, y_{i+1}^B, \ldots, y_n^B]^\top \) the productions of agents other than \( i \). We denote by \( \mathbf{p}^A = [p_1^A, \ldots, p_n^A]^\top \) and \( \mathbf{p}^B = [p_1^B, \ldots, p_n^B]^\top \) the vectors of prices.

The utility of agent \( i \) follows a linear-quadratic function:

\[
\begin{align*}
\partial u_i(y_i, y_{-i}) &= p_i^Ay_i^A - \frac{1}{2}c_i^A(y_i^A)^2 + p_i^By_i^B - \frac{1}{2}c_i^B(y_i^B)^2 \\
&= \begin{cases} \\
\text{Proceeds from } A & \text{ Proceeds from } B \\
- \sum_{j=1}^n G_{ij}^A y_j^A & - \sum_{j=1}^n G_{ij}^B y_j^B \\
\text{Global network effect in activity } A & \text{Global network effect in activity } B \\
\delta_i^A G_{ij}^A y_j^A & + \delta_i^B G_{ij}^B y_j^B \\
\text{Local network effect from activity } A & \text{Local network effect from activity } B \\
\mu_i^A & + \mu_i^B \\
\text{Marginal effect of global competition for activity } A & \text{Marginal effect of global competition for activity } B \\
\beta_i^A & \beta_i^B \\
\text{Agent-specific coupling parameter between activities } A & \text{interaction between activity } A \text{ and activity } B \\
\end{cases}
\end{align*}
\]

**TABLE I: Notation**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_i^A )</td>
<td>Price for activity ( A ) in dollar per unit of production (($/m^3))</td>
</tr>
<tr>
<td>( p_i^B )</td>
<td>Price for activity ( B ) in dollar per unit of production (($/m^3))</td>
</tr>
<tr>
<td>( c_i^A )</td>
<td>Unit cost of activity ( A ) (($/m^3))</td>
</tr>
<tr>
<td>( c_i^B )</td>
<td>Unit cost of activity ( B ) (($/m^3))</td>
</tr>
<tr>
<td>( \delta_i^A )</td>
<td>Marginal local network effect for activity ( A ) (($/m^3))</td>
</tr>
<tr>
<td>( \delta_i^B )</td>
<td>Marginal local network effect for activity ( B ) (($/m^3))</td>
</tr>
<tr>
<td>( \mu_i^A )</td>
<td>Marginal effect of global competition for activity ( A ) (($/m^3))</td>
</tr>
<tr>
<td>( \mu_i^B )</td>
<td>Marginal effect of global competition for activity ( B ) (($/m^3))</td>
</tr>
<tr>
<td>( G^A )</td>
<td>Adjacency matrix of the network underlying activity ( A )</td>
</tr>
<tr>
<td>( G^B )</td>
<td>Adjacency matrix of the network underlying activity ( B )</td>
</tr>
<tr>
<td>( \beta_i )</td>
<td>Agent-specific coupling parameter between activities (($/m^3))</td>
</tr>
</tbody>
</table>

Note that the agents’ equilibrium activity levels can be shaped by other factors. For example, one can consider identical prices across agents (\( p_i^A = p_i^A \) and \( p_i^B = p_i^B \)) or cost of production (\( c_i^A = c_i^A \) and \( c_i^B = c_i^B \)). Our results (Theorem 2) can be used to evaluate the impact of such factors, although our main focus is on the impact of coupling introduced by \( \beta_i \) on \( y_i^A, y_i^B \) under heterogeneous network structures underlying \( A \) and \( B \).

**B. Existence and uniqueness**

It is important to recall from [9] that a simpler network game with a single activity admits a potential function. We establish analogous result for our more general game of two coupled activities. We know from [14] that for a game with continuous and twice-differentiable utility function \( u_i \), there exists a potential function if and only if \( \frac{\partial u_i(y, y_{-i})}{\partial y_i} = \frac{\partial u_i(y, y_{-i})}{\partial y_{-i}} \) for all \( i, j \in \{1, \ldots, n\} \). Our game \( \Gamma \) satisfies these conditions since

\[
\frac{\partial u_i(y, y_{-i})}{\partial y_i} = \frac{\partial u_i(y, y_{-i})}{\partial y_{-i}} = \delta_i^A G_{ij}^A, \quad \frac{\partial u_i(y, y_{-i})}{\partial y_{-i}} = \delta_i^B G_{ij}^B.
\]
potential function of the game $\Gamma$ and $\partial u_i(y_i,y_{-i}) = \frac{\partial u_i(y_i,y_{-i})}{\partial y_i}$, and $\partial u_i(y_i,y_{-i}) = \frac{\partial u_i(y_i,y_{-i})}{\partial y_{-i}} = 0$, thus $\Gamma$ admits a potential function. We consider the potential function of the game $\Gamma$:
\[
\Phi(y_i, y_{-i}) = p_i \sum_{i=1}^n y_i - c_i y_i + \mu_i y_i^2/2 - D_i \sum_{j=1}^n y_j y_i - \mu_i \sum_{j=1}^n y_j y_i^2 - 2 \sum_{i=1}^n y_i y_i^2.
\]

The proof can be found in Appendix A. The authors in [11] offer a more general result on the equilibrium existence in the negative of the Hessian of potential function of the game. In an economic context, it means that the definiteness of the Hessian of potential function of the game only involve own agent’s production. Otherwise, a large network effect requires that the network effect must be small enough compared to own individual concavity, i.e., quadratic terms (here costs) that only involve own agent’s production. Otherwise, a large enough local network effect would compromise the positive definiteness of $-H$. Notice that the global competition term does not show this effect because they are counted negatively in the utility function. In an economic context, it means that the influence of direct neighbors should not exceed the effect of one’s own production cost and global competition net the coupling effect between the two activities.

In proving Theorem 1, we simply leveraged the fact that our game is a potential game and concluded existence and uniqueness of the Nash equilibrium based on positive definiteness of the Hessian of potential function of the game. On the other hand, authors of [12, 16] (building on results of [15]), offer a more general result on the equilibrium existence and uniqueness based on variational inequality framework provided Assumption 4 and strong monotonicity of operator $V$ holds.

Indeed, for the game $\Gamma$, Theorem 1 can be also derived through the lens of the variational inequality with set $Y = Y_1 \times \cdots \times Y_n$ and operator $V : Y \rightarrow \mathbb{R}^{n \times 2}$, $V(y) := -[\nabla y_i u_i(y_i,y_{-i})]_{i=1}^n$.

C. Equilibrium characterization

For each agent $i$, the first-order conditions of the game are given by:
\[
\begin{align*}
0 &= p_i - c_i y_i + \beta_i y_i - \mu_i y_i - \mu_i \sum_{j=1}^n y_j + \delta^A \sum_{j=1}^n y_j y_i = 0, \\
0 &= p_i - c_i y_i + \beta_i y_i - \mu_i y_i - \mu_i \sum_{j=1}^n y_j + \delta^B \sum_{j=1}^n y_j y_i = 0.
\end{align*}
\]

Following the notation in Section III-B, we can express these conditions in matrix form:
\[
\begin{bmatrix}
D & -\beta \\
-\beta & Q
\end{bmatrix}
\begin{bmatrix}
y^A \\
y^B
\end{bmatrix}
= \begin{bmatrix}
p^A \\
p^B
\end{bmatrix}.
\]

If the existence and uniqueness condition in Theorem 1 holds, systems (3) is invertible (refer to [17, Lemma 1]):
\[
\begin{bmatrix}
y^A \\
y^B
\end{bmatrix}
= \begin{bmatrix}
D & -\beta \\
-\beta & Q
\end{bmatrix}^{-1}
\begin{bmatrix}
p^A \\
p^B
\end{bmatrix}.
\]

Let us proceed with the following notation:
\[
\begin{align*}
\hat{G}^A &= -c_i + \mu_i J + \delta^A \hat{G}^A, \\
\hat{G}^B &= -B_j + \mu_i J + \delta^B \hat{G}^B, \\
L_A &= \frac{1}{c_i + \mu_i (I - \hat{G}^A)^{-1}}, \\
L_B &= \frac{1}{c_i + \mu_i (I - \hat{G}^B)^{-1}}.
\end{align*}
\]

Here, $(c^A + \mu^A)L^A$ and $(c^B + \mu^B)L^B$ are the so-called Leontief matrices [13]. Let us also define $B_A = (c^A + \mu^A)L_A I = (I - \hat{G}^A)^{-1} I$ and $B_B = (c^B + \mu^B)L_B I = (I - \hat{G}^B)^{-1} I$. Following Definition 1, the vectors $B_A$ and $B_B$ can be interpreted as Bonacich centralities for the networks with adjacency matrices $G^A$ and $G^B$ respectively.

We also define the weighted Bonacich centralities $B_A = (c^A + \mu^A)L_A p^A = (I - \hat{G}^A)^{-1} p^A$ and $B_B = (c^B + \mu^B)L_B p^B = (I - \hat{G}^B)^{-1} p^B$, where the weights are the corresponding price vectors for activities $A$ and $B$, i.e., $p^A = [p^{A1}, \ldots, p^{An}]^T$ and $p^B = [p^{B1}, \ldots, p^{Bn}]^T$.

For ease of notation and simplify equations, let us adopt the following notation:
\[
\begin{align*}
\hat{B}_A &= \frac{1}{c_i + \mu_i} \hat{B}_A, \\
\hat{B}_B &= \frac{1}{c_i + \mu_i} \hat{B}_B.
\end{align*}
\]

Rewriting (6), the system to be solved is the following:
\[
\begin{bmatrix}
y^A \\
y^B
\end{bmatrix}
= \begin{bmatrix}
L_A^{-1} & -\beta \\
-\beta & L_B^{-1}
\end{bmatrix}
\begin{bmatrix}
p^A \\
p^B
\end{bmatrix} = \begin{bmatrix}
Z_1 \\
Z_2
\end{bmatrix} \begin{bmatrix}
p^A \\
p^B
\end{bmatrix}.
\]
where by the inversion formulae of block diagonal matrices, $Z_1, Z_2, Z_3$ and $Z_4$ are given as follows:

\[
Z_1 = [L^{-1}_A - \beta L_B \beta]^{-1}, \\
Z_2 = L_A \beta [L_B - \beta L_A \beta]^{-1}, \\
Z_3 = L_B \beta [L_B - \beta L_A \beta]^{-1}, \\
Z_4 = [L_A^{-1} - \beta L_B \beta]^{-1},
\]

Further, in order to simplify notation, we introduce coefficients $K_1, K_2, K_3$ and $K_4$, such that $Z_1 = K_1 L_A, Z_2 = K_2 L_B, Z_3 = K_3 L_A$ and $Z_4 = K_4 L_B$, where we define

\[
K_1 = [I - L_A \beta L_B \beta]^{-1}, \\
K_2 = [I - L_B \beta L_A \beta]^{-1}, \\
K_3 = L_B \beta [I - L_A \beta L_B \beta]^{-1} = L_B \beta K_1, \\
K_2 = L_A \beta [I - L_B \beta L_A \beta]^{-1} = L_A \beta K_2.
\]

We can now characterize the equilibrium of the game $\Gamma$.

**Theorem 2.** Assume $\min(c^1 + \mu^1, c^2 + \mu^2) - \max_i \{\|\beta_i\|\} > \max(\delta^1 \lambda_{\max}(G^A), \delta^2 \lambda_{\max}(G^B))$, then the game $\Gamma$ admits a unique Nash equilibrium given by:

\[
y^A = K_1 B_A + K_2 B_B, \\
y^B = K_3 B_A + K_4 B_B.
\]

For the special case of uniform prices across agents (i.e., $p^A = p^A I$ and $p^B = p^B I$):

\[
y^A = p^A K_1 B_A + p^B K_2 B_B, \\
y^B = p^A K_3 B_A + p^B K_4 B_B.
\]

To interpret this result, we recall that [11] consider a simpler game in which the activities $A$ and $B$ have the same structure $G^A = G^B$ and $\delta^A = \delta^B$. In their case, they show that, in equilibrium activities can be expressed as a sum of two Bonacich centralities. In our game, networks $G^A$ and $G^B$ are different and $\delta^A \neq \delta^B$. Theorem 2 provides that the equilibrium for the game $\Gamma$ - if unique exists, can be given by a linear combination of $B_A$ and $B_B$, i.e., the vectors of Bonacich centralities of the networks with adjacency matrices $G^A$ and $G^B$ respectively.

Furthermore, if there is no coupling between activities (i.e. $\beta = 0$), then $y^A = B_A$ and $y^B = B_B$. Indeed, in such a case, the equilibrium of each activity can be analyzed independently, and we are left with the result from [8, 9, 10, 13]. When $\beta \neq 0$, the vectors of Bonacich centralities $B_A$ and $B_B$ only depend on the corresponding network for $A$ and $B$ respectively and are weighted by the weight matrices $K_1, K_2, K_3$ and $K_4$. We can interpret these weight matrices coefficient by going back to matrices $Z_1, Z_2, Z_3$ and $Z_4$.

Let us discuss the general case of agent-specific prices $p^A$ and $p^B$. Consider a marginal increase in the price for agent 1, that is $p^{A}_{1}$ increases by $+1$ for agent 1. Then, by Theorem 2 Equation (5), the new equilibrium is given by

\[
y_{new}^A = Z_1 p^A + Z_2 [1, 0, \ldots, 0]^T, y_{new}^B = Z_3 p^B + Z_4 [1, 0, \ldots, 0]^T.
\]
between the local network parameters delta $\delta^A$, $\delta^B$ and the coupling parameter $\beta$. 

The density of the network structures also affects the production level at the equilibrium. For positive network effects $\delta^A$, $\delta^B$ and a positive coupling effect $\beta$, a denser network in $A$ (e.g. complete network where all agents are connected) increases the effort made by the agents compared to a sparser network (e.g., empty network where the degree of every node is 0). This is expected as a positive network effect $\delta^A > 0$ means that the feedback from neighbors affect individual agents positively. When $\delta^A < 0$, the feedback from neighbors is negative. In such a case, a denser network in activity $A$ will negatively affect the equilibrium activity levels.

The density of the other activity $B$ also impacts the effect in activity $A$, because of the coupling $\beta$ between activities. In particular, assuming $\beta > 0$, given a network structure in activity $A$, if the network effects $\delta^A$ and $\delta^B$ are positive, the production in $A$ will be higher when the network structure for activity $B$ is denser. For instance, in Figure 1 when the network structure in $A$ is complete and $\delta^A, \delta^B > 0$ (respectively $\delta^A, \delta^B < 0$), the production in $A$ is higher (respectively smaller) when the network structure in activity $B$ is complete compared to an empty network.

These effects are ambiguous if $\beta$ and $(\delta^A, \delta^B)$ are of different signs. For instance, if $\beta < 0$ and $(\delta^A > 0, \delta^B > 0)$, a denser network in $B$ increases the utility through the local network term $\delta^B \sum_{j=1}^n G^B_{ij} y_i^A y_j^B$, but this also has a negative impact because of the cross-activity term $\beta y_i^A y_i^B$. If $\beta > 0$ (and $\delta^A < 0, \delta^B < 0$), a denser network in $B$ decreases the utility through the local network term $\delta^B \sum_{j=1}^n G^B_{ij} y_i^A y_j^B$, thus pushing $y_i^A$ downwards, while the cross-activity term $\beta y_i^A y_i^B$ tends to push it upwards.

Ceteris paribus, the analysis in the coupling parameter $\beta$ is more straightforward. A negative $\beta$ parameter affects the equilibrium level negatively while it has a positive effect when it is positive ($\beta > 0$). Overall, for fixed local parameters $\delta^A$, $\delta^B$ and fixed network structures, an increase in $\beta$ increases the level of effort at the equilibrium.

In summary, a trade-off arises between the local network effect $(\delta^A, \delta^B)$ and the coupling parameter $(\beta)$, namely for a given production level, a decrease (resp. increase) in the local network terms $\delta^A \sum_{j=1}^n G^A_{ij} y_i^A y_j^A + \delta^B \sum_{j=1}^n G^B_{ij} y_i^A y_j^B$ can be traded against an increase (resp. decrease) in the coupling term $\beta y_i^A y_i^B$. This trade-off is illustrated in Figure 2.

V. CONCLUSION

We study a linear-quadratic game with two activities and heterogeneous network structures for these activities. We find a sufficient condition for the existence and uniqueness of a Nash equilibrium and we show that this condition can be equivalently studied through the potential game property of this game or using variational inequality results. Furthermore, we prove that the equilibrium can be written as a linear combination of Bonacich centralities. As future work, an econometric analysis based on this model should be conducted in order to empirically test our theoretical model and in particular estimate the effect of the local network terms. In the context of illegal logging, such an analysis would enable us to estimate the effect of the spatial layout of concessions.

APPENDIX

A. Proof of Theorem 7

In order to simplify the notation, we show the proof for Theorem 7 assuming $\mu^A = \mu^B = \mu$. $(c^A + \mu) I + \mu J - \delta^A G^A$ is positive definite if and only if:

$$
\begin{align*}
\mathbf{a}^T ((c^A + \mu) I + \mu J - \delta^A G^A) \mathbf{a} > 0, \forall \mathbf{a} \neq 0, \\
(c^A + \mu) \mathbf{a}^T \mathbf{a} > \delta^A \mathbf{a}^T G^A \mathbf{a}, \forall \mathbf{a} \neq 0, \\
(c^A + \mu) \frac{\mathbf{a}^T \mathbf{a}}{\mathbf{a}^T G^A \mathbf{a}} > \delta^A, \forall \mathbf{a} \neq 0.
\end{align*}
$$

In particular, this must be true for any $\mathbf{a} \neq 0$. We have:

$$
\begin{align*}
(c^A + \mu) \min_{\mathbf{a} \neq 0} \frac{\mathbf{a}^T \mathbf{a}}{\mathbf{a}^T G^A \mathbf{a}} + \mu \min_{\mathbf{a} \neq 0} \frac{\mathbf{a}^T \mathbf{a}}{\mathbf{a}^T G^A \mathbf{a}} > \delta^A, \\
(c^A + \mu) \min_{\mathbf{a} \neq 0} \frac{\mathbf{a}^T \mathbf{a}}{\mathbf{a}^T G^A \mathbf{a}} + \mu \min_{\mathbf{a} \neq 0} \frac{\mathbf{a}^T \mathbf{a}}{\mathbf{a}^T G^A \mathbf{a}} > \delta^A, \\
(c^A + \mu) \frac{1}{\max_{\mathbf{a} \neq 0} \frac{\mathbf{a}^T \mathbf{a}}{\mathbf{a}^T G^A \mathbf{a}}} + \mu \min_{\mathbf{a} \neq 0} \frac{\mathbf{a}^T \mathbf{a}}{\mathbf{a}^T G^A \mathbf{a}} > \delta^A.
\end{align*}
$$

By the Rayleigh-Ritz theorem, we have $\max_{\mathbf{a} \neq 0} \frac{\mathbf{a}^T G^A \mathbf{a}}{\mathbf{a}^T \mathbf{a}} = \lambda_{\max}(G^A)$, where $\lambda_{\max}(G^A)$ is the largest eigenvalue of $G^A$ and $\min_{\mathbf{a} \neq 0} \frac{\mathbf{a}^T G^A \mathbf{a}}{\mathbf{a}^T \mathbf{a}} = \lambda_{\min}(G^A)$ where $\lambda_{\min}(G^A)$ is the smallest eigenvalue of $G^A$. Hence, the condition for the positive definiteness property is:

$$
\delta^A < (c^A + \mu) \lambda_{\max}(G^A) + \mu \frac{\lambda_{\min}(J)}{\lambda_{\max}(G^A)}.
$$

Fig. 1: Individual production (in log) of activity $A$ as a function of the network parameter $\delta_A$. We assume $\delta^A = \delta^B$, $\mu^A = \mu^B = 0.01$, $\beta_i = \beta = 0.4$, $\forall i$.

Fig. 2: Curves of iso-production in activity $A$ for an individual agent at the equilibrium. The network structure is complete for both $G^A$ and $G^B$, $n = 20$, $\mu = 0.01$. 

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The eigenvalues of $J$ are $n$ with multiplicity 1 and 0 with multiplicity $n - 1$. We thus deduce that $(c^A + \mu I + \mu J - \delta^A G^A)$ is positive definite if and only if

$$c^A + \mu > \delta^A \lambda_{\text{max}}(G^A).$$ \quad (10)

Let us now find a sufficient condition for the positive definiteness of the Schur complement of $((c^A + \mu)I + \mu J - \delta^A G^A)$. The Schur complement of $((c^A + \mu)I + \mu J - \delta^A G^A)$ is

$$((c^A + \mu)I + \mu J - \delta^A G^A) - \beta((c^B + \mu)I + \mu J - \delta^B G^B)^{-1} \beta.$$ 

We thus want to show a condition such that

$$((c^A + \mu)I + \mu J - \delta^A G^A) - \beta((c^B + \mu)I + \mu J - \delta^B G^B)^{-1} \beta > 0,$$

$$(c^A + \mu)I + \mu J - \delta^A G^A - (\max_i \{\beta_i\})^2((c^B + \mu)I + \mu J - \delta^B G^B)^{-1} > 0,$$ \quad (11)

$$(c^A + \mu)I + \mu J - \delta^A G^A > (\max_i \{\beta_i\})^2 I.$$ 

Then $\forall \mathbf{a}^\top \not= 0$, we have:

$$\max_{\mathbf{a}^\top \not= 0} \frac{((c^A + \mu)I + \mu J - \delta^B G^B)((c^A + \mu)I + \mu J - \delta^A G^A)\mathbf{a}}{\mathbf{a}^\top \mathbf{a}} > (\max_i \{\beta_i\})^2.$$ \quad (12)

Furthermore, for $A$ and $B$ two positive definite matrices, we have $\lambda_{\text{max}}(AB) < \lambda_{\text{max}}(A)\lambda_{\text{max}}(B)$. Therefore, we have:

$$\max_{\mathbf{a}^\top \not= 0} \frac{((c^A + \mu)I + \mu J - \delta^A G^A)\mathbf{a}}{\mathbf{a}^\top \mathbf{a}} > (\max_i \{\beta_i\})^2,$$

$$\max_{\mathbf{x} \in \{A, B\}} \left( \max_{\mathbf{a}^\top \not= 0} \frac{((c^A + \mu)I + \mu J - \delta^A G^A)\mathbf{a}}{\mathbf{a}^\top \mathbf{a}} \right)^2 > (\max_i \{\beta_i\})^2. $$ \quad (13)

Quantities on both sides of the inequality are positive scalars and we can take the square root of these quantities.

$$\max_{\mathbf{x} \in \{A, B\}} \max_{\mathbf{a}^\top \not= 0} \frac{((c^A + \mu)I + \mu J - \delta^A G^A)\mathbf{a}}{\mathbf{a}^\top \mathbf{a}} > \max_i \{\beta_i\}. $$ \quad (14)

Since $\lambda_{\text{max}}(J) = n$, a sufficient condition is:

$$\max_{\mathbf{x} \in \{A, B\}} (c^X + \mu) - \max_{\mathbf{a}^\top \not= 0} \frac{\mathbf{x}^\top \mathbf{G}^X \mathbf{a}}{\mathbf{a}^\top \mathbf{a}} > \max_i \{\beta_i\},$$

$$\max(\delta^A \lambda_{\text{max}}(G^A), \delta^B \lambda_{\text{max}}(G^B)) < - \max_i \{\beta_i\} + \max(c^X + \mu; c^B + \mu),$$ \quad (15)

and we deduce that the condition for existence and uniqueness of a Nash equilibrium is

$$\min(c^A + \mu - \max_i \{\beta_i\}; c^B + \mu - \max_i \{\beta_i\}) > \max(\delta^A \lambda_{\text{max}}(G^A), \delta^B \lambda_{\text{max}}(G^B)).$$ \quad (16)