Distributed Consensus-based Kalman Filter under Limited Communication

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***Abstract*— In this work, we consider a distributed estimation problem in a communication-constrained environment. To address the limited communication challenge, we present a fully distributed Kalman filtering algorithm in which each agent shares a compressed version of its estimated state information with its neighboring nodes. In the proposed algorithm, we explicitly compute the estimation error covariances of each node in a distributed manner based on the consensus filter using the compressed estimates. An intuitive finding is that for a specific mid-tread quantization function, compared with the uncompressed distributed Kalman consensus filter, the state estimates obtained with the quantized Kalman consensus filter are significantly similar; however, the estimation error covariances are noticeably different. We validate the theoretical results using simulations.**

I. Introduction

As autonomy is integrated into more Cyber Physical Systems (CPS), control and decision-making increasingly depend on sensors being able to aggregate and process data. With CPSs becoming larger, the need to decentralize or distribute decision-making and actuation action, and to manage the available communication resources becomes important. A key element in enabling distributed decision-making is distributed estimation. Of interest in this work are distributed estimation algorithms for communication-constrained environments where agents are to manage the limited bandwidth for communication and coordination. Communication-efficient distributed estimation algorithms have broad applications in many areas including guidance, navigation, and control of autonomous vehicles and other CPS such as power systems.

Bayesian filters have been very widely studied with the Kalman filter being a widely used variant in different applications. Estimation (filtering) theory finds applications in many diverse fields: communications, radar, navigation, biomedical engineering, and finance, among others [1–3]. Cooperative decentralized and distributed filters have been studied in the different contexts including [4] Until recently, the question of how to compute the estimation error covariances and cross-covariances associated with obtained estimates in a distributed manner had not been studied in depth [5].

Communication-efficient filtering methods have been proposed in the literature [6–8]. A major difference between existing work and ours is that we consider the computation of the estimation error covariances in a distributed manner, based on the recent work in [5]. In addition, the work in the two-part series [8] considered the quantization of agent observations, as opposed to their state estimates.

This work focuses on the fully distributed Kalman filter operating in an environment with limited communication. Specifically, we have assume that the bandwidth available for communication and coordination is very limited and only compressed pieces of information can be exchanged between neighboring nodes as they cooperatively estimate the quantity of interest. While there are different ways of considering communication efficiency in such settings, including event-triggered communication schemes and sparsifying the vectors broadcast, we focus on quantizing or compressing the information being broadcast to a few bits for communication.

We obtain an explicit expression for the gain, consensus, and estimation error covariance matrices, computed as a function of the quantized information received (or quantization function). In comparison to the unconstrained communication case, this work focuses on the first aspect which is sending compress or contacts estimates. In summary, we show that when quantized information is communicated in the Kalman consensus scheme, the resulting estimation error covariance matrices do not significantly change in comparison with the uncompressed version. We note that the particular characterization of the quantizer used can affect the state estimates obtained and the resulting estimation error covariances.

The rest of the paper is organized as follows: In Section II, we introduce the problem and proposed algorithm. The main results are presented in Sections III and IV, where we respectively summarize computations of the estimation error covariances and Kalman gains. We present simulations of the theoretical results in Section V and follow with concluding remarks in Section VI.

II. Problem Formulation

We consider a linear discrete-time system whose dynamics at time-step k are expressed by (1)

$$x_k = A_{k-1}x_{k-1} + B_{k-1}u_{k-1} + G_{k-1}w_{k-1} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the system state, $u_k \in \mathbb{R}^m$ is the system input, $w_k \in \mathbb{R}^n$ is the process noise, $A_k \in \mathbb{R}^{n\times n}$ is the state matrix, $B_k \in \mathbb{R}^{n\times m}$ is the input matrix, and $G_k \in \mathbb{R}^{n\times n}$ is the process noise matrix. We assume that the process noise is white Gaussian with a zero mean, that is $\mathbb{E}[w_k] = 0$, and a known covariance $Q_k = \mathbb{E}[w_k w_k^T]$. Our objective is to estimate x_k without having access to the

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state in (1). To do this, we consider a heterogeneous set of N spatially distributed nodes. Using measurements possibly different from other nodes, each node i computes an estimate of the state via the following relation

$$z_{i,k} = H_{i,k}x_k + v_{i,k}, \tag{2}$$

where $z_{i,k} \in \mathbb{R}^{p_i}$ are the measurements for each node i, $v_{i,k} \in \mathbb{R}^{p_i}$ is the measurement noise, and $H_{i,k} \in \mathbb{R}^{p_i \times n}$ is the measurement noise matrix. Similar to the process noise, we assume that the measurement noise is white Gaussian with a zero mean, that is $\mathbb{E}[v_k] = 0$, and a known covariance $R_{i,k} = \mathbb{E}[v_{i,k}v_{i,k}^T]$. Different values of $H_{i,k}$ and $R_{i,k}$ produce different observations for node i, in which case, we talk about a heterogeneous set of nodes.

We consider an estimation scenario in which each node cooperates with neighboring nodes by sharing its state estimate with other nodes through a communication network. This network is represented as strongly connected graph $\mathcal{G} = (\mathcal{O},\mathcal{E})$ where $\mathcal{O} \triangleq \{o_1, \ldots, o_{\mathrm{N}}\}$ is the previously described set of N nodes and $\mathcal{E} \subseteq \mathcal{O} \times \mathcal{O}$ is a set of directed edges among the nodes. We refer to (o_i, o_j) as an edge from o_i to o_j. We define the adjacency matrix $\mathcal{A} = \{a_{ij}\}$ as a N×N matrix such that $a_{ij} = 1$ if $(o_j, o_i) \in \mathcal{E}$, and $a_{ij} = 0$, otherwise. The adjacency matrix defines how the information is shared among the nodes and is allowed to change between time steps and we assume it is always known to all the nodes. The neighborhood of each node i at time-step k, defined as $\mathcal{N}_{i,k}$ is the subset of $\mathcal{O}$ such that $\mathcal{N}_{i,k} = \{o_j \in \mathcal{O} | (o_j, o_i) \in \mathcal{E}\}$. We do not consider node i to be a member of $\mathcal{N}_{i,k}$.

The fundamental problem addressed in this paper is to estimate (1) using heterogeneous observations given by (2) under limited communication. We consider a Distributed Kalman Filter based on consensus [4, 9–11]. In these filtering approaches, the estimation update step includes a term based on the estimation from the node's neighbors. Recently, a Distributed Consensus Kalman Filter (DCKF) that permits each node i to use distinctive weighting matrices for each neighbor and calculate the estimation error cross-covariances for the entire network was introduced in [5] with update equations

$$\hat{x}_{i,k}^- = A_{k-1}\hat{x}_{i,k-1}^+ + B_{k-1}u_{k-1} \tag{3}$$

$$\hat{x}_{i,k}^+ = \hat{x}_{i,k}^- + K_{i,k}(z_{i,k} - H_{i,k}\hat{x}_{i,k}^-) + \sum_{j \in \mathcal{N}_{i,k}} M_{ij,k}(\hat{x}_{j,k}^- - \hat{x}_{i,k}^-), \tag{4}$$

where $\hat{x}_{i,k}^+ \in \mathbb{R}^n$ and $\hat{x}_{i,k}^- \in \mathbb{R}^n$ are, respectively, the predicted and updated state estimates, $K_{i,k}$ is the Kalman gain, and $M_{ij,k}$ is the consensus matrix that weights each neighbor's contribution. The closed form expressions of $K_{i,k}$ and $M_{ij,k}$ can be found in [5]. Our main contribution is to revisit the DCKF when the communication among nodes is limited. Under our framework, the nodes can share information about their estimates with other nodes using limited communication bandwidth. To address this constraint, we employ a quantizer $Qu(\cdot)$ resulting in new expressions for the DCKF update equations

$$\hat{x}_{i,k}^+ = \hat{x}_{i,k}^- + K_{i,k}(z_{i,k} - H_{i,k}\hat{x}_{i,k}^-) + \sum_{j \in \mathcal{N}_{i,k}} M_{ij,k}(Qu(\hat{x}_{j,k}^-) - \hat{x}_{i,k}^-). \tag{5}$$

It is important to notice that the prediction equation does not change, since the exchange of information occurs only during the update step. We use the quantizer over each variable of $\hat{x}_{i,k}^-$ independently. In the rest of the paper, we analyze how quantization of exchanged state information affects the distributed estimation process, and propose a new Quantized Distributed Consensus Kalman Filter (Q-DCKF) that represents a conservative solution to the described problem.

III. Calculation of the Cross-Covariance Matrices

The calculation of the estimation error cross-covariance matrices represents a relevant problem for Distributed Kalman Filters (DKF) because the knowledge of the cross-correlations between a node and its neighbors cannot be always assumed. Based on Equation (5), we need to consider not only the estimation error, but also the quantization error. Let $\eta_{i,k}^- = \hat{x}_{i,k}^- - x_k$ be the prediction error, $\eta_{i,k}^+ = \hat{x}_{i,k}^+ - x_k$ the update error, and $e_{i,k} = Qu(\hat{x}_{i,k}^-) - \hat{x}_{i,k}^-$ the quantization error. Similar to [11], the update error can be expressed as a function of the prediction and quantization errors by replacing Equations (2) and (5) into the definition of $\eta_{i,k}^+$ to obtain (6).

$$\eta_{i,k}^+ = F_{i,k}\eta_{i,k}^- + K_{i,k}v_{i,k} + \sum_{j \in \mathcal{N}_{i,k}} M_{ij,k}(Q(\eta_{j,k}^-) - \eta_{i,k}^-, \tag{6}$$

where $F_{i,k} = I - K_{i,k}H_{i,k}$ and $Qu(\eta_{j,k}^-) = \eta_{j,k}^- + e_{j,k}$. In the same way, the prediction error can be expressed as a function of the update error by substituting Equations (2) and (3) into the definition of $\eta_{i,k}^-$ to obtain (7).

$$\eta_{i,k}^- = A_{k-1}\hat{\eta}_{i,k-1}^+ - G_{k-1}w_{k-1}. \tag{7}$$

For any pair of nodes (i, j), it is necessary to consider not only the cross-covariance between the estimation errors; that is, the prediction error cross-covariance $P_{ij,k}^{Q-}$ and the update error cross-covariance $P_{ij,k}^{Q+}$, but also the cross-correlation between quantization errors $J_{ij,k}$ and the cross-covariance between the estimation error and the quantization error $U_{ij,k}$.

$$P_{ij,k}^{Q-} = \mathbb{E}[\eta_{i,k}^-\eta_{j,k}^{-T}] \tag{8}$$

$$P_{ij,k}^{Q+} = \mathbb{E}[\eta_{i,k}^+\eta_{j,k}^{+T}] \tag{9}$$

$$J_{ij,k} = \mathbb{E}[e_{i,k}e_{j,k}^T] \tag{10}$$

$$U_{ij,k} = \mathbb{E}[\eta_{i,k}^-e_{j,k}^T] \tag{11}$$

The cross-covariance and cross-correlation matrices are important because $tr(P_{ij,k}^{Q+})$ represents and important measure of the estimation error. Under this consideration, we introduce an expression for $P_{ij,k}^{Q+}$ and $P_{ij,k}^{Q-}$ in the following lemma.

Lemma 1. *Consider a system with dynamics represented by* (1) *and* (2)*. For any sensor nodes $i, j \in \mathcal{O}$, the estimation error cross-covariances of Equations* (3) *and* (5)*, are given by*

$$P^-_{ij,k} = A_{k-1}P^+_{ij,k-1}A^T_{k-1} + G_{k-1}Q_{k-1}G^T_{k-1} \tag{12a}$$

$$\begin{aligned} P^+_{ij,k} =& F_{i,k}P^{Q-}_{ij,k}F^T_{j,k} + K_{i,k}R_{ij,k}K^T_{j,k} \\ &- F_{i,k}\sum_{s\in\mathcal{N}_{j,k}}(P^{Q-}_{ij,k} - P^{Q-}_{is,k})M^T_{js,k} \\ &- \sum_{r\in\mathcal{N}_{i,k}} M_{ir,k}(P^{Q-}_{ij,k} - P^{Q-}_{rj,k})F^T_{i,k} \\ &+ \sum_{r\in\mathcal{N}_{i,k}}\sum_{s\in\mathcal{N}_{j,k}} M_{ir,k}(P^{Q-}_{ij,k} - P^{Q-}_{is,k} \\ &- P^{Q-}_{rj,k} + P^{Q-}_{rs,k})M^T_{js,k} \end{aligned} \tag{12b}$$

$$\begin{aligned} \zeta_{ij,k} =& - F_{i,k}\sum_{s\in\mathcal{N}_{j,k}} U_{is,k}M^T_{js,k} \\ &- \sum_{r\in\mathcal{N}_{i,k}} M_{ir,k}U^T_{jr,k}F^T_{j,k} \\ &+ \sum_{r\in\mathcal{N}_{i,k}}\sum_{s\in\mathcal{N}_{j,k}} M_{ir,k}(U_{rs,k} + U^T_{sr,k} \\ &- U^T_{jr,k} - U_{is,k} + J_{rs,k})M^T_{js,k} \end{aligned} \tag{12c}$$

$$P^{Q+}_{ij,k} = P^+_{ij,k} + \zeta_{ij,k} \tag{12d}$$

$$P^{Q-}_{ij,k} = P^-_{ij,k} + A_{k-1}\zeta_{ij,k-1}A^T_{k-1} \tag{12e}$$

where $R_{ij,k} = \mathbb{E}[v_{i,k}v^T_{j,k}]$.

Proof: This is a direct consequence of substituting Equation (6) into Equation (9) and (7) into Equation (8).

Before focusing on the influence of the quantization process, it is important to note that for node i to compute $P^{Q+}_{ij,k}$, it needs not only the prediction cross-covariance with node j but the prediction cross-covariance with the neighbors of node j. Olfati *et al.* first recognized this dependence on *second-hop neighbors* in [10], and Howard *et al.* [5] proposed a way to overcome this limitation if node i has access to $R_{j,k}$ and $H_{j,k}$. From (12d) and (12e), the calculation of $P^{Q-}_{ij,k}$ and $P^{Q+}_{ij,k}$ does not depend on the measurements or the state estimations. Therefore, each node can calculate by itself the expressions of $P^-_{ml,k}$ and $P^+_{ml,k}$ $\forall\ l, m \in \mathcal{O}$ as long as we assume that $R_{lm} = 0\ \forall\ l \neq m$. By expressing (12d) as the sum of two terms, we can partially describe the influence of the quantization process by looking at $\zeta_{ij,k}$. Indeed, the expressions for $P^-_{ij,k}$ and $P^+_{ij,k}$ in (12a) and (12b) are equal to those found in [5] except for the dependence by $P^+_{ij,k}$ on $P^{Q-}_{ij,k}$, that is, on the quantized equivalent of $P^-_{ij,k}$.

Furthermore, the influence of quantization is defined by the cross-correlations defined in (10) and (11) of node i with the neighbors of j, node j with the neighbors of node i, and the neighbors of node i with the neighbors of j. The calculation of U_{ij} and J_{ij} represents an open problem due to obstacles like the mentioned dependency on second neighbors, but also due to the dependency on the nature of the employed quantizer. In general, finding a recursive expression for the quantization error cannot be done in the same way it was done for the estimation error in (6). Despite this obstacle, the implementation of a Q-DCKF remains feasible because each node can still calculate (12a) and (12b) by itself if we use $P^-_{ij,k}$ instead of $P^{Q-}_{ij,k}$ in (12b). In that case, we can understand $\zeta_{ij,k}$ in (12d) and $A_{k-1}\zeta_{ij,k}A^T_{k-1}$ in (12e) as the propagation of the quantization error through the estimation cross-covariance. In the next section, we use this perspective to determine the parameters that define the proposed Q-DCKF.

IV. Kalman and Consensus Gains for the Quantized DCKF

The Kalman gain and the Consensus gain parameters are necessary in the update step of the Q-DCKF. Despite the limitation over $\zeta_{ij,k}$ described in the previous section, a bound to $P^{Q+}_{ij,k}$ can be found for the case where $i = j$. The results are presented in the following lemma.

Lemma 2. *Consider a system with dynamics determined by* (1) *and* (2)*. For any $i \in \mathcal{O}$, and a scalar $\sigma_{i,k} > 0$ there exists an upper bound to the estimation error covariance of* (3) *and* (5) *such that $P^{Q+}_{ii,k} \leq \hat{P}^{Q+}_{ii,k}$ where*

$$\begin{aligned} \hat{P}^{Q+}_{ii,k} =& (1+\sigma_{i,k})\ [F_{i,k}P^{Q-}_{ii,k}F^T_{i,k} \\ &- F_{i,k}\sum_{j\in\mathcal{N}_{i,k}}(P^{Q-}_{ii,k} - P^{Q-}_{ij,k})M^T_{ij,k} \\ &- \sum_{j\in\mathcal{N}_{i,k}} M_{ij,k}(P^{Q-}_{ii,k} - P^{Q-}_{ji,k})F^T_{i,k} \\ &+ \sum_{r\in\mathcal{N}_{i,k}}\sum_{s\in\mathcal{N}_{i,k}} M_{ir,k}(P^{Q-}_{rs,k} - P^{Q-}_{ri,k} \\ &- P^{Q-}_{is,k} + P^{Q-}_{ii,k})M^T_{is,k}] \\ &+ K_{i,k}R_{ii,k}K^T_{i,k} \\ &+ (1+\sigma^{-1}_{i,k})\sum_{r\in\mathcal{N}_{i,k}}\sum_{s\in\mathcal{N}_{i,k}} M_{ir,k}J_{rs,k}M^T_{is,k}. \end{aligned} \tag{13}$$

Proof: This is a consequence of rearranging the terms in (12d) appropriately and the inequality $\mathbb{E}[xy^T + yx^T] \leq \mathbb{E}[\sigma xx^T + \sigma^{-1}yy^T]$ [12, 13] for a scalar $\sigma > 0$.

Note that unlike $P^{Q+}_{ij,k}$, $\hat{P}^{Q+}_{ii,k}$ does not depend on the calculations of $U_{ij,k}$ between estimation and quantization error, but only on the cross covariances due to quantization error $J_{ij,k}$, whose value depends on the nature of the quantizers employed. This bound also depends on a scalar $\sigma_{i,k}$, whose optimal determination represents on a non-convex optimization problem. The determination of the Kalman gain $K_{i,k}$ and the Consensus gain $M_{ij,k}$ are equivalent to solving the following problem

$$K_{i,k}, M_{ij,k} = \arg\min_{K_{i,k}, M_{ij,k}} tr(\hat{P}^{Q+}_{ii,k}) \tag{14}$$

The solution of (14) for $K_{i,k}$ is presented in Lemma 3.

Lemma 3. *Consider a system with dynamics* (1) *and* (2) *and an estimator* (3) *and* (5)*. For any $i \in \mathcal{O}$, and a scalar*

$\sigma_{i,k} > 0$, the Kalman gain $K_{i,k}$ that solves (14) *is equal to*

$$K_{i,k} = [(1+\sigma_{k,i})P^{Q-}_{ii,k} - \sum_{j\in\mathcal{N}_i} M_{ij,k}(P^{Q-}_{ii,k} - P^{Q-}_{ji,k})] \times H^T_{i,k}[(1+\sigma_{k,i})H_{i,k}P^{Q-}_{ii,k}H^T_{i,k} + R_{ii,k}]^{-1}. \quad (15)$$

Proof: The proof is presented in Appendix A

It is important to note that despite the fact that $K_{i,k}$ does not depend directly on the cross-covariances defined in (10), it has an indirect dependence based on the chosen form of the consensus gains $M_{ij,k}$. Moreover, $K_{i,k}$ is a function of $M_{ij,k}$ and $\sigma_{i,k}$, so these parameters can have any value or form. We define $M_{ij,k}$ for some $\alpha_{i,k} \in \mathbb{R}$ in the same way as in [5], that is

$$M_{ij,k} = \alpha_{i,k} A^T_{k-1}\sqrt{R^{-1}_{ii,k}R^{-1}_{jj,k}}. \quad (16)$$

This expression allows each node to calculate the value of $P^-_{ij,k}$ and $P^+_{ij,k}$ without depending on information of second neighbors besides $R_{m,k}$ and $H_{m,k}$ $\forall\, m \in \mathcal{O}$. The expression of (15) can now be expressed as $K_{i,k} = \Psi_{i1} - \alpha_{i,k}\Psi_{i2}$ where

$$\Psi_{i1} = (1+\sigma_{k,i})P^{Q-}_{ii,k}H^T_{i,k}[(1+\sigma_{k,i})H_{i,k}P^{Q-}_{ii,k}H^T_{i,k} + R_{ii,k}]^{-1} \quad (17)$$

$$\Psi_{i2} = (1+\sigma_{k,i})[\sum_{j\in\mathcal{N}_i} A^T_{k-1}\sqrt{R^{-1}_{i,k}R^{-1}_{j,k}}(P^{Q-}_{ii,k} - (P^{Q-}_{ij,k})^T)]H^T_{i,k}[(1+\sigma_{k,i})H_{i,k}P^{Q-}_{ii,k}H^T_{i,k} + R_{ii,k}]^{-1} \quad (18)$$

We use this notation of $K_{i,k}$ to determine the optimal value of $M_{ij,k}$. Given the form considered in (16), the problem (14) translates into finding the optimal value for $\alpha_{i,k}$, whose result is presented in lemma 4. Compared to the non-quantized case [5], the expressions of the Kalman and Consensus gains differ mainly on the dependence on $\sigma_{i,k}$ and $J_{ij,k}$. These two differences characterize the influence of the quantization process of the estimate among nodes. The Q-DKCF procedure is summarized in Algorithm 1.

Algorithm 1 Quantized Distributed Kalman Consensus Filter for Agent i

1: Initialize $\hat{x}^+_{i,0}$, $P^+_{mm,0} \leftarrow I^{n\times n}$, $P^+_{ml,0} \leftarrow 0^{n\times n}$ $\forall\, m,l \in \mathcal{N}$, $m \neq l$
2: **for** each timestep k **do**
3: Predict $\hat{x}^-_{i,k}$ according to (3)
4: Predict $P^-_{ml,k}$ $\forall\, m,l \in \mathcal{N}$according to (12a)
5: Calculate $M_{ml,k}$ $\forall\, m,l \in \mathcal{N}$ according to (16)
6: Calculate $K_{m,k}$ $\forall\, m \in \mathcal{N}$ according to (15)
7: Update $x^+_{i,k}$ according to (5)
8: Update $P^+_{ml,k}$ $\forall\, m,l \in \mathcal{N}$ according to (12b)

Lemma 4. *Consider a system with dynamics* (1)*, sensor measurements* (2) *and an estimator defined by* (3) *and* (5)*. For any $i \in \mathcal{O}$, and a scalar $\sigma_{i,k} > 0$, the Consensus gain $M_{ij,k}$ that solves* (14) *is equal to $\alpha_{i,k} = -tr(\Psi_{i4})/tr(\Psi_{i5})$ where*

$$\Psi_{i3} = I - \Psi_{i1}H_{i,k} \quad (19)$$

$$\Psi_{i4} = (1+\sigma_{i,k})[\Psi_{i2}H_{i,k}P^{Q-}_{ii,k}\Psi^T_{i3} + \Psi_{i3}P^{Q-}_{ii,k}H^T_{i,k}\Psi^T_{i2} - \Psi_{i3}\sum_{j\in\mathcal{N}_{i,k}}(P^{Q-}_{ii,k} - P^{Q-}_{ij,k})\sqrt{R^{-1}_{ii,k}R^{-1}_{jj,k}}^T A_{k-1} - \sum_{j\in\mathcal{N}_{i,k}} A^T_{k-1}\sqrt{R^{-1}_{ii,k}R^{-1}_{jj,k}}(P^{Q-}_{ii,k} - P^{Q-}_{ji,k})\Psi^T_{i3}] - \Psi_{i2}R_{ii,k}\Psi^T_{i1} - \Psi_{i1}R_{ii,k}\Psi^T_{i2} \quad (20)$$

$$\Psi_{i5} = 2(1+\sigma_{i,k})[\Psi_{i2}H_{i,k}P^{Q-}_{ii,k}H^T_{i,k}\Psi^T_{i2} + \sum_{r\in\mathcal{N}_{i,k}}\sum_{s\in\mathcal{N}_{i,k}} A^T_{k-1}\sqrt{R^{-1}_{ii,k}R^{-1}_{rr,k}}(P^{Q-}_{rs,k} - P^{Q-}_{ri,k} - P^{Q-}_{is,k} + P^{Q-}_{ii,k})\sqrt{R^{-1}_{ii,k}R^{-1}_{ss,k}}^T A_{k-1} - \Psi_{i2}H_{i,k}\sum_{j\in\mathcal{N}_{i,k}}(P^{Q-}_{ii,k} - P^{Q-}_{ij,k})\sqrt{R^{-1}_{ii,k}R^{-1}_{jj,k}}^T A_{k-1} - \sum_{j\in\mathcal{N}_{i,k}} A^T_{k-1}\sqrt{R^{-1}_{ii,k}R^{-1}_{jj,k}}(P^{Q-}_{ii,k} - P^{Q-}_{ji,k})H^T_{i,k}\Psi^T_{i2}] + 2\Psi_{i2}R_{ii,k}\Psi^T_{i2} + 2(1+\sigma^{-1}_{k,i})\sum_{r\in\mathcal{N}_{i,k}}\sum_{s\in\mathcal{N}_{i,k}} A^T_{k-1}\sqrt{R^{-1}_{ii,k}R^{-1}_{jj,k}}J_{rs,k}\sqrt{R^{-1}_{ii,k}R^{-1}_{jj,k}}^T A_{k-1} \quad (21)$$

(22)

Proof: The proof is presented in Appendix B

V. Numerical Simulation

While the results presented above are for a generic quantization function, the numerical simulation presented assumes the quantizer Qu to be a uniform mid-tread quantizer [14–16].

A. Uniform Quantizer

Uniform quantizers are such that the quantization step size is a constant, independent from the value of the quantizer input. The mid-treat quantizer we consider is defined as:

$$Qu(x) = \begin{cases} \delta - 2^{\lambda-1}q & \forall x \in (-\infty, \delta - (2^\lambda+1)\frac{q}{2}] \\ \delta + \text{sgn}(x-\delta)q\lfloor\frac{|x-\delta|}{q} + \frac{1}{2}\rfloor & \forall x \in (\delta - (2^\lambda+1)\frac{q}{2}, \\ & \quad \delta + (2^\lambda-1)\frac{q}{2}) \\ \delta + (2^{\lambda-1}-1)q & \forall x \in [\delta + (2^\lambda-1)\frac{q}{2}, +\infty), \end{cases} \quad (23)$$

where $q \in \mathbb{R}$ is the quantization step size, δ is the mid-point of the quantization interval, and λ is the number of bits disposable. The length of the quantization interval l is related to the quantization step size by $l = 2^n q$. Figure 1 illustrates an example of such quantizer for a small value of λ. Based on (23), q is the main parameter that defines the uniform quantizer. In consequence, we expect that the statistical analysis of the quantization error $e_{i,k}$, and the

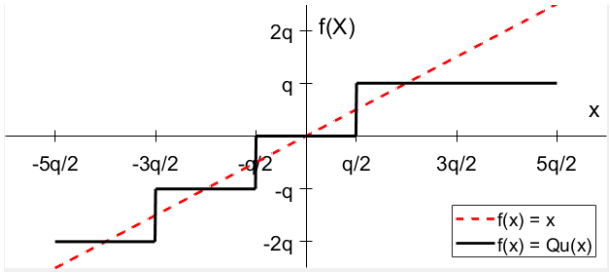

Fig. 1: Uniform mid-raiser quantizer for $\delta = 0$, and $n = 2$

corresponding cross-covariance defined in (10) to directly depend on q. This connection is clear as we assume that the probability distribution function of each variable in $e_{i,k}$ resembles that of independent uniformly distributed noise between $-q/2$ and $q/2$ [17]. Under this assumption, the value of J_{rs} is given by

$$J_{rs,k} = \begin{cases} q^2/12\ I^{n\times n} & r = s \\ 0^{n\times n} & r \neq s. \end{cases} \tag{24}$$

Although not made explicit in Algorithm 1, the calculation cost of the the cross-covariances can be reduced by considering that $P^+_{ij,k} = P^{+T}_{ji,k}$ and $P^-_{ij,k} = P^{-T}_{ji,k}$. This fact means that each node only need to calculate the lower or upper triangle of $P^-_k = \{P^-_{ij,k}\}$ and $P^+_k = \{P^+_{ij,k}\}$. Howard *et al.* [5] locally distribute the calculation of P^+_k, but that is not possible in our case due to the limitation on the communication channel.

B. Simulation Parameters

For the simulations, we use the Linear Time-Varying System considered in [5] to have a reference of the influence of the quantization process. This system has no input, that is $u_k = 0\,\forall\, k$. Also the matrices G_k, Q_k, and H_k are equal to the identity I with the appropriate dimensions. Nevertheless, A_k changes with each time-step, and $R_{i,k}$ changes for each node. More specifically,

$$A_k = \begin{bmatrix} 1 + 0.025\sin(0.3k) & -0.015 \\ 0.015 & 1 + 0.05\sin(0.5k) \end{bmatrix} \tag{25}$$

$$R_{i,k} = 5e^i \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{26}$$

The system and corresponding measurements for N=5 are shown in Figure 2. Three different communication network topologies, shown in Figure 3, are considered. In Figure 3 (a), for example, node 1 receives information from node 2 and shares information with node 5. From Figure 2, it is clear that the measurements are better for nodes with a small index like o_1 because of how the measurement noise covariances are defined in Equation (26). The noise, for example, affects the measurement of node 5 the most. First, we present the results for the non-quantized DCKF. In Figure 4, we see that the error, measured by the $trace(P^+_{ii,k})$ and the Root Mean Square Error (RMSE) converge. Every time there is a change in the topology, there is an abrupt change in $trace(P^+_{ii,k})$.

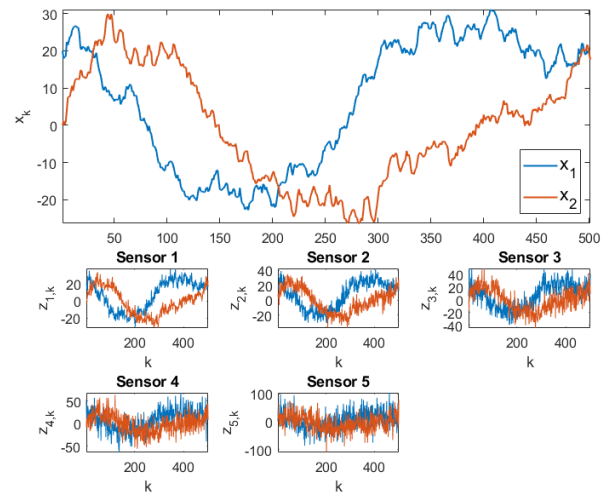

Fig. 2: Value of x_k to estimate and heterogeneous measurements $z_{i,k}$ for a system with parameters (25) and (26)

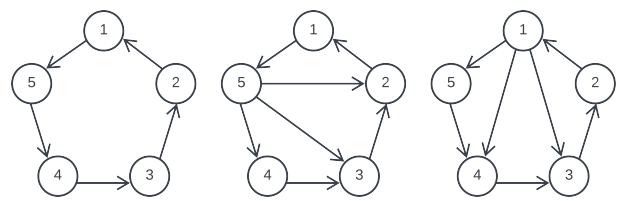

Fig. 3: Topologies considered [5] for the communication network from left to right for a) $k <= 200$, b) $200 < k <= 400$, and c) $k > 400$ for the system given by (25) and (26)

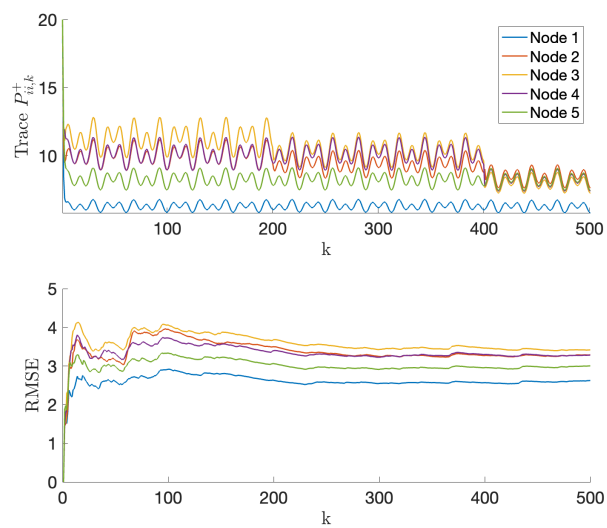

Fig. 4: Distributed Consensus Kalman Filter for the system considered in (25) and (26) under a communication network with changing topologies

For the quantized case, the quantizer $Qu(\cdot)$ is designed in a way that ensures the number of bits is high enough

to avoid truncation errors; that is, a case where the input to the quantizer being out of the quantization interval. We consider that all the available bits are used to represent the integer part and none are used to represent the fractional part. In consequence, the minimum value that can be shared

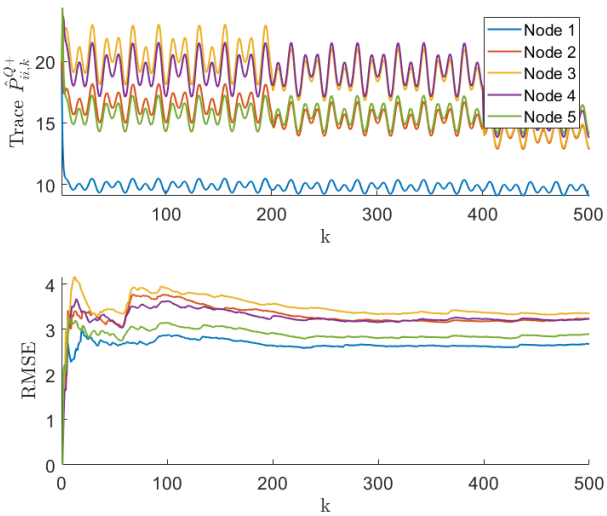

Fig. 5: Proposed Quantized Distributed Consensus Kalman Filter ($q = 1$, $\sigma_{i,k} = 1$) for the system considered in (25) and (26) under a communication network with changing topologies

among agents is $q = 1$. The results for the proposed Q-DCKF are presented in Figure 5. While a noticeable change in $trace(\hat{P}^{Q+}_{ii,k})$ compared to $trace(P^{+}_{ii,k})$ is observed, it is expected due to the dependence on $\sigma_{i,k}$ and the fact that $trace(\hat{P}^{Q+}_{ii,k})$ actually represents an upper bound. On the other hand, the change in the RMSE is much smaller, which suggest that the expressions from (15) to (21) and the design of the filter are appropriate. Moreover, the influence of the communication is preserved under the quantized approach. The performance of some nodes improves based on the information received from their neighbors. Node 5, for example, has the worst noise covariace, yet has smaller error than node 3 because it receives information from node 1, which has a lower noise covariance.

VI. Conclusion

In this paper, we present a novel Quantized Distributed Consensus Kalman Filter. General closed form expressions for the Kalman gain and the Consensus have been obtained based on an upper bound to the covariance that is dependent on the nature of the employed quantizers. It is important to notice that different type of quantizer could be considered for each node as long as the cross-correlation between the quantization errors is known. The proposed algorithm has been tested using an uniform quantizer. The results are positive in the sense that the estimation error is kept under acceptable value. Future work could be focused on the stability analysis and the optimal design of the scalar parameters that define the upper bound to the nodes' covariances.

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APPENDIX A

Proof of Lemma 3: We calculate the partial derivative of $tr(\hat{P}^{+}_{ii,k})$ with respect to $K_{i,k}$

$$\begin{aligned}\frac{\partial tr(\hat{P}^{Q+}_{ii,k})}{\partial K_{i,k}} =& 2(1+\sigma_{k,i})[-P^{Q-}_{ii,k}H^T_{i,k} + K_{i,k}H_{i,k} \\ & P^{Q-}_{ii,k}H^T_{i,k} + 2\sum_{j\in\mathcal{N}_i} M_{ij,k}(P^{Q-}_{ii,k} \\ & - (P^{Q-}_{ij,k})^T)H^T_{i,k}] + 2K_{i,k}R_{ii,k}\end{aligned} \tag{27}$$

Equating this to zero, and isolating $K_{i,k}$, we obtain the expressions presented in (14).

APPENDIX B

Proof of Lemma 4 We calculate the partial derivative of $tr(\hat{P}^{+}_{ii,k})$ with respect to $\alpha_{i,k}$.

$$\begin{aligned}\frac{\partial tr(\hat{P}^{Q+}_{ii,k})}{\partial \alpha_{i,k}} =& (1+\sigma_{i,k})[\Psi_{i2}H_{i,k}P^{-}_{ii,k}F^T_{i,k} + F_{i,k}P^{Q-}_{ii,k} \\ & H^T_{i,k}\Psi^T_{i2} + \sum_{r\in\mathcal{N}_{i,k}}\sum_{s\in\mathcal{N}_{i,k}} A^T_{k-1}\sqrt{R^{-1}_{ii,k}R^{-1}_{rr,k}} \\ & (P^{Q-}_{rs,k} - P^{Q-}_{ri,k} - P^{Q-}_{is,k} + P^{Q-}_{ii,k})M^T_{is,k} \\ & + \sum_{r\in\mathcal{N}_{i,k}}\sum_{s\in\mathcal{N}_{i,k}} M_{ir,k}(P^{Q-}_{rs,k} - P^{Q-}_{ri,k} \\ & - P^{Q-}_{is,k} + P^{Q-}_{ii,k})\sqrt{R^{-1}_{ii,k}R^{-1}_{ss,k}}^T A_{k-1} \\ & - F_{i,k}\sum_{j\in\mathcal{N}_{i,k}}(P^{Q-}_{ii,k} - P^{Q-}_{ij,k})\sqrt{R^{-1}_{ii,k}R^{-1}_{jj,k}}^T \\ & A_{k-1} - \Psi_{i2}H_{i,k}\sum_{j\in\mathcal{N}_{i,k}}(P^{Q-}_{ii,k} - P^{Q-}_{ij,k})M^T_{ij,k} \\ & - \sum_{j\in\mathcal{N}_{i,k}} A^T_{k-1}\sqrt{R^{-1}_{ii,k}R^{-1}_{jj,k}}(P^{Q-}_{ii,k} \\ & - P^{Q-}_{ji,k})F^T_{i,k} - \sum_{j\in\mathcal{N}_{i,k}} M_{ij,k}(P^{Q-}_{ii,k} - P^{Q-}_{ji,k}) \\ & H^T_{i,k}\Psi^T_{i2}] + 2(1+\sigma^{-1}_{i,k})\sum_{j\in\mathcal{N}_{i,k}}\sum_{s\in\mathcal{N}_{i,k}} A^T_{k-1} \\ & \sqrt{R^{-1}_{ii,k}R^{-1}_{jj,k}}e_{r,k}e^T_{s,k}\sqrt{R^{-1}_{ii,k}R^{-1}_{jj,k}}^T A_{k-1} \\ & - \Psi_{i2}R_{ii,k}K^T_{i,k} - K_{i,k}R_{ii,k}\Psi^T_{i2} \\ & = 0\end{aligned} \tag{28}$$

Equating this to zero, and isolating $\alpha_{i,k}$, we obtain the expressions presented in Equations (17) to (21).